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MEROMORPHIC-STARLIKENESS-PRESERVING PROPERTIES FOR AN INTEGRAL OPERATOR: SOME NEW RESULTS AND REMARKS

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Abstract

In recent years, there has been considerable interest in classes of functions meromorphic and their possible connections with the applied mathematics. In two previous papers [1] and [2], we shown two properties about the meromorphic-starlikeness preserving for an integral operator. Starting these results in the present paper, we obtain some new properties and remarks. A survey of this theory and applications can be found in [5].

1. Introduction

Let $\sum_k$ be the class of meromorphic functions f in the unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ having the form

$$f(z) = \frac{1}{z} + a_k z^k + \cdots, \quad z \in \mathcal{U}.$$

A function $f \in \sum = \sum_0$ is called starlike, if

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$$\Re\left[-\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in \mathcal{U}.$$

Let denote by Σ_k^* the class of starlike functions in Σ_k. Let A_n denote the class of functions

$$g(z) = z + a_{n+1}z^{n+1} + \dots, \quad z \in \mathcal{U},\ n \geq 1,$$

that are analytic in $\mathcal{U}$.

For $g \in A_{k+1}$, with $g(z)/z \neq 0$ in $\mathcal{U}$, and $c > 0$, let us define the integral operators

$$K_{g,c}(f)(z) = \frac{c}{g^{c+1}(z)}\int_0^z f(t)g^c(t)dt, \quad z \in \mathcal{U},\ f \in \Sigma, \tag{1}$$

$$I_{g,c},\ J_{g,c},\ K_{g,c},\ \text{and } L_{g,c,\gamma} : \Sigma \to \Sigma, \tag{2}$$

by the following equations:

(1) $$I_{g,c}(f)(z) = \frac{c}{g^{c+1}(z)}\int_0^z f(t)g^c(t)g'(t)dt, \quad z \in \mathcal{U},\ f \in \Sigma;$$

(2) $$J_{g,c}(f)(z) = \frac{c}{g^{c+1}(z)}\int_0^z \frac{zf(t)g^{c+1}(t)}{t}dt, \quad z \in \mathcal{U},\ f \in \Sigma;$$

(3) $$K_{g,c}(f)(z) = \frac{c}{g^{c+1}(z)}\int_0^z f(t)g^c(t)dt, \quad z \in \mathcal{U},\ f \in \Sigma;$$

(4) $$L_{g,c,\gamma}(f)(z) = \frac{c}{g^{c+1}(z)}\int_0^z f(t)g^c(t)e^{\gamma t^p}dt, \quad z \in \mathcal{U},\ f \in \Sigma.$$

In [1] and [2], the authors found sufficient conditions on c and g so that

$$I_{g,c}(\Sigma_k^*) \subset \Sigma_k^*, \quad J_{g,c}(\Sigma_k^*) \subset \Sigma_k^*, \text{ and } K_{g,c}(\Sigma_k^*) \subset \Sigma_k^*.$$

2. Preliminaries

In order to obtain our main result of the previous papers [1] and [2], we will use the following definitions and lemmas:

If f and g are analytic functions in $\mathcal{U}$ and g is univalent, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if $f(0) = g(0)$ and if $f(\mathcal{U}) \subset g(\mathcal{U})$.

The analytic function f, with $f(0) = 0$ and $f'(0) \neq 0$ is starlike in $\mathcal{U}$ (i.e., f is univalent in $\mathcal{U}$ and $f(\mathcal{U})$ is starlike with respect to the origin), if and only if

$$\Re \frac{zf'(z)}{f(z)} > 0 \text{ for all } z \in \mathcal{U}.$$

Lemma 1. *Let h be starlike in $\mathcal{U}$ and let $p(z) = 1 + p_n z^n + \dots$ be analytic in $\mathcal{U}$. If*

$$\frac{zp'(z)}{p(z)} \prec h(z),$$

then $p \prec q$, where

$$q(z) = \exp \frac{1}{n} \int_0^z \frac{h(t)}{t} dt.$$

The proof of this lemma was given by Suffridge in [6].

Lemma 2. *Let the function $\psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$ satisfy the condition*

$$\Re\psi[ix, y; z] \leq 0,$$

for all $z \in \mathcal{U}$ and for all real x and $y \leq -n(1 + x^2)/2$.

If $p(z) = 1 + p_n z^n + \dots$ is analytic in $\mathcal{U}$ and

$$\Re\psi[p(z), zp'(z); z] > 0 \text{ for all } z \in \mathcal{U},$$

then $\Re p(z) > 0$ in $\mathcal{U}$.

Lemma 3. *Let B and C be two complex functions in $\mathcal{U}$ satisfying*

$$|\Im C(z)| \le n \Re B(z), \quad z \in \mathcal{U},$$

where n is a positive integer. If $p(z) = 1 + p_n z^n + \ldots$ is analytic in $\mathcal{U}$ and

$$\Re[B(z)zp'(z) + C(z)p(z)] > 0 \quad \text{for } z \in \mathcal{U},$$

then $\Re p(z) > 0$ in $\mathcal{U}$.

The proof of the last two lemmas are simple applications of the more general theory of differential subordinations, due to Miller and Mocanu. In [1] and [2], we proved the following results:

Theorem 1. *Let $c > 0$ and let k be a positive integer. If $g \in A_{k+1}$ and $g(z)/z \neq 0$ in $\mathcal{U}$ and if $G(z) = zg'(z)/g(z)$ satisfies*

$$\left|\Im\left[(c+1)g'(z) - \frac{g(z)}{z}\right]\right| \le (k+1)\Re\frac{g(z)}{z}, \quad z \in \mathcal{U}, \tag{3}$$

$$\Re G(z) > \frac{2}{(k+1)(c+1)+2}, \quad z \in \mathcal{U}, \tag{4}$$

$$\begin{aligned}(c+1)\{\Im[zG'(z) - 2G(z)] + 2\operatorname{Re} G(z)\Im G(z)\}^2 \\ \le \{[(k+1)(c+1)+2]\Re G(z) - 2\} \\ \times\left\{\left[k+1+2(c+1)|G(z)|^2\right]\Re G(z) + 2(c+1)\left[\Re zG'(z)\overline{G(z)} - |G(z)|^2\right]\right\},\end{aligned} \tag{5}$$

then $K_{g,c}(\Sigma_k^*) \subset \Sigma_k^*$, where the integral operator $K_{g,c}$ is defined by (1).

Theorem 2. *Let $\gamma \in \mathbb{C}$, $c > 0$, and let p and k be positive integers. If $g \in A_{k+1}$ is starlike and $g(z)/z \neq 0$ in $\mathcal{U}$ and if $G(z) = zg'(z)/g(z)$ satisfies*

$$\left|\Im\left[(c+1)g'(z) - \frac{g(z)}{z}\right]e^{-\gamma z^p}\right| \le (k+1)\Re\frac{g(z)}{z}e^{-\gamma z^p}, \quad z \in \mathcal{U}, \tag{6}$$

$$[2 + (k+1)(c+1)]\Re G(z) > 2[1 + p\operatorname{Re}\gamma z^p], \quad z \in \mathcal{U}, \tag{7}$$

$$(c+1)\left[\Im zG'(z) - 2\Im G(z)\Re(1 - G(z) + \gamma pz^{p})\right]^{2}$$

$$\leq \{[2 + (k+1)(c+1)]\Re G(z) - 2[1 + p\Re\gamma z^{p}]\}$$

$$\times\left\{\left[k+1+2(c+1)|G(z)|^{2}\right]\Re G(z) + 2(c+1)\Re zG'(z)\overline{G(z)} - 2(c+1)|G(z)|^{2}(1 + p\Re\gamma z^{p})\right\}, \tag{8}$$

then $L_{g,c,\gamma}(\Sigma_k^*) \subset \Sigma_l^*$, *where the integral operator* $L_{g,c,\gamma}$ *is defined by* (5) *and* $l = \min\{p-1, k\}$.

3. Main Result

Corollary 1. *If* $|\lambda| \leq 1/15 = 0.066\ldots$ *and if* K *is the integral operator defined by* $F = K(f)$, *where*

$$F(z) = \frac{1}{z^{2}(1+\lambda z^{2})^{2}}\int_{0}^{z} f(t)t(1+\lambda t^{2})dt,$$

then $K(\Sigma_1^*) \subset \Sigma_1^*$.

Proof. We let in Theorem 1 $c = 1$, $k = 1$, and $g(z) = z(1+\lambda z^{2})$. Then

$$G(z) = 1 + \frac{2\lambda z^{2}}{1+\lambda z^{2}}, \quad z \in \mathcal{U}.$$

Condition (3) becomes

$$|\Im(1 + 5\lambda z^{2})| \leq 2\Re(1+\lambda z^{2}). \tag{9}$$

If we put $\lambda z^{2} = \zeta = \rho e^{i\theta}$, from (9), we easily obtain

$$5\rho|\sin\theta| \leq 2 + 2\rho\cos\theta.$$

It is easy to show that this last inequality holds for all real θ if $\rho \leq 2/7 = 0.2857\ldots.$ Using the same notations, condition (5) becomes

$$\Re\left(1 + \frac{2\zeta}{1 + \zeta}\right) > \frac{1}{3},$$

i.e.,

$$\frac{3\rho^2 + 4\rho\cos\theta + 1}{\rho^2 + 2\rho\cos\theta + 1} > \frac{1}{3},$$

which can be rewritten as

$$4\rho^2 + 5\rho\cos\theta + 1 > 0.$$

It is easy to show that this inequality holds for all real θ if $\rho \leq 1/4 = 0.2.$ After some calculations, condition (5) becomes $f(\rho) \geq 0,$ where

$$\begin{aligned} f(\rho) = {} & 120\rho^8 + 390\rho^7\cos\theta + (533 + 2041\cos^2\theta)\rho^6 \\ & + (2612 + 1651\cos^2\theta)\rho^5\cos\theta + (68\cos^4\theta + 1210\cos^2\theta + 524)\rho^4 \\ & + (890 + 316\cos^2\theta)\rho^3\cos\theta + (117\cos^2\theta + 97)\rho^2 + 22\rho\cos\theta + 2. \end{aligned}$$

It is easy to show that this last inequality holds for all real θ if $\rho \leq 1/15.$ Thus, we conclude that for every $\theta \in \mathbb{R}$ and for $\rho \leq 1/15 = 0.066...$ conditions (3), (4), and (5) are satisfied. Hence, by applying Theorem 1, we deduce the result stated in the corollary.

Remark 1. If we let $\gamma = 0,$ by applying Theorem 2, we obtain the result from [2].

Remark 2. If we let $c = k = p - 1 = 1,$ $g(z) = z\exp\frac{\lambda z^2}{2},$ and $\gamma = -\lambda/2,$ then $G(z) = 1 + \lambda z^2$ and for $|\lambda| < 1,$ we have immediately that $\Re G(z) > 0$ in $\mathcal{U}.$ Hence, g is starlike in $\mathcal{U}$ for $|\lambda| < 1.$

Let $\lambda z^2 = \rho e^{i\theta},$ $0 < \rho < 1,$ $\theta \in \mathbb{R}$ and let $\tau = \rho\sin\theta \in (-1, 1).$ Condition (6) is equivalent to

$$|2\rho\sin(\theta + \tau) + \sin\tau| \leq 2\cos\tau.$$

It is easy to show that this inequality holds for all $\theta \in \mathbb{R}$ and $\rho \leq (\sqrt{2}-1)/2$. Condition (7) is equivalent to

$$4(1 + \rho \cos\theta) > 0,$$

which is true for all $\rho \in (0, 1)$.

Condition (8) is equivalent to

$$\rho^4 \sin^2 2\theta - 4\rho^3 \cos^3 \theta - 3\rho^2(2\cos^2 \theta + 1) - 6\rho \cos\theta - 1 \leq 0.$$

It is easy to show that this last inequality holds for all $\rho \leq (\sqrt{2}-1)/2$. Hence, by applying Theorem 2, we deduce the following result:

Corollary 2. *If* $\lambda \in \mathbb{C}$ *with* $|\lambda| \leq (\sqrt{2}-1)/2 = 0.2071\ldots$ *and if L is the integral operator defined by* $F = L(f)$, *where*

$$F(z) = \frac{1}{z^2 e^{\lambda z^2}} \int_0^z t f(t) dt,$$

then $L(\Sigma_1^*) \subset \Sigma_1^*$.

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